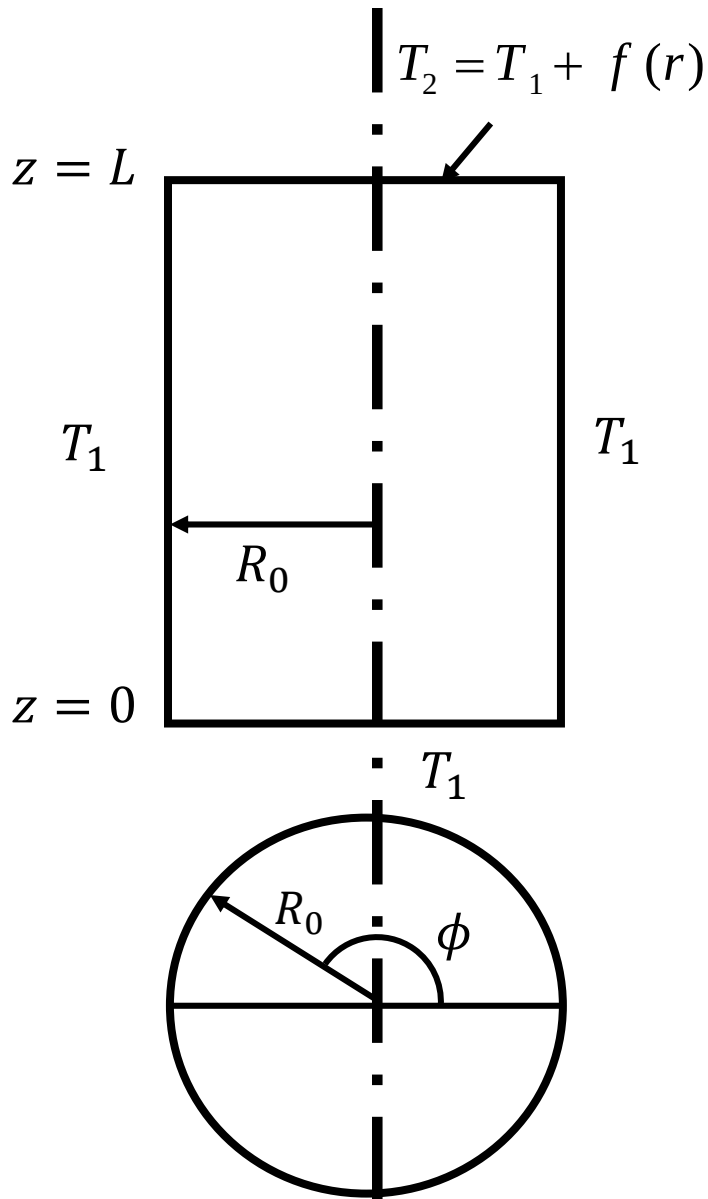


Heat and Mass transfer

2-D Heat Conduction in a Finite Cylinder

2-D Heat conduction in a Finite Cylinder



➤ Boundary Conditions:

$$\text{At } r = R_0, \quad T = T_1, \quad 0 < z < L$$

$$\text{At } z = 0, \quad T = T_1, \quad 0 < r < R$$

$$\text{At } z = L, \quad T = T_1 + f(r), \quad 0 \leq r \leq R_0$$

$$\text{At } r = 0, \quad T = \text{finite}$$

➤ Governing Equation:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

2-D Heat conduction in a Finite Cylinder

- For steady state

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Let us put $\theta = T - T_1$ and consider ϕ -symmetry. Then we can write

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} = 0 \quad \text{Governing Eqn}$$

- The boundary conditions are

$$\text{at } r = R_0, \quad \theta = 0, \quad 0 < z < L$$

$$\text{at } r = 0, \quad \theta \text{ is finite}, \quad 0 < z < L$$

$$\text{at } z = 0, \quad \theta = 0, \quad 0 < r < R_0$$

$$\text{at } z = L, \quad \theta = f(r), \quad 0 \leq r \leq R_0$$

2-D Heat conduction in a Finite Cylinder

- We now apply the separation of variables technique by assuming that the desired solution can be expressed as the product of two functions, one of which depends only on r while the other depends only on z

$$\theta(r, Z) = R(r) Z(z)$$

$$\text{Then } \frac{\partial^2 \theta}{\partial r^2} = R''(r) Z(z), \quad \frac{\partial \theta}{\partial r} = R'(r) Z(z)$$

$$\text{and } \frac{\partial^2 \theta}{\partial z^2} = R(r) Z''(z)$$

- The GE becomes

$$\frac{1}{R(r)} \left[R''(r) + \frac{1}{r} R'(r) \right] + \frac{Z''(z)}{Z(z)} = 0$$

$$\text{or, } -\frac{1}{R(r)} \left[R''(r) + \frac{1}{r} R'(r) \right] = \frac{Z''(z)}{Z(z)} = \lambda^2$$

2-D Heat conduction in a Finite Cylinder

We get $Z''(z) - \lambda^2 Z(z) = 0$

$$Z = A e^{-\lambda z} + B e^{\lambda z} \quad (1)$$

$$R''(r) + \frac{1}{r} R'(r) + \lambda^2 R = 0$$

The form matches with

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = 0$$

which is Bessel's equation of zero order

➤ The solutions are

$$R = C J_0(\lambda r) + D Y_0(\lambda r) \quad (2)$$

$J_0(\lambda r)$ is the Bessel function of the first kind of order zero

$Y_0(\lambda r)$ is the Bessel function of the second kind of order zero

Bessel's functions

➤ Bessel's equation of order n: $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$

➤ Solutions are:

- Bessel's function of first kind:

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m}$$

- Bessel's function of second kind:

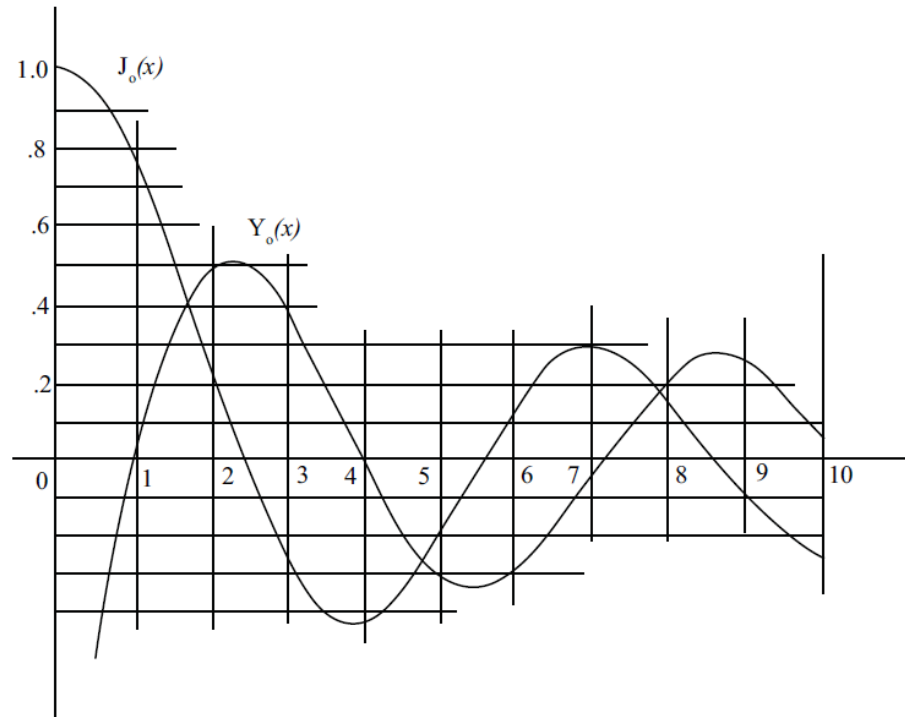
$$Y_n(x) = \frac{2}{\pi} \left\{ \ln \left(\frac{x}{2} \right) + \gamma \right\} J_n(x) - \frac{1}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2} \right)^{2m-n}$$

where $\gamma = 0.5772156$ is an Euler's constant

Bessel's functions

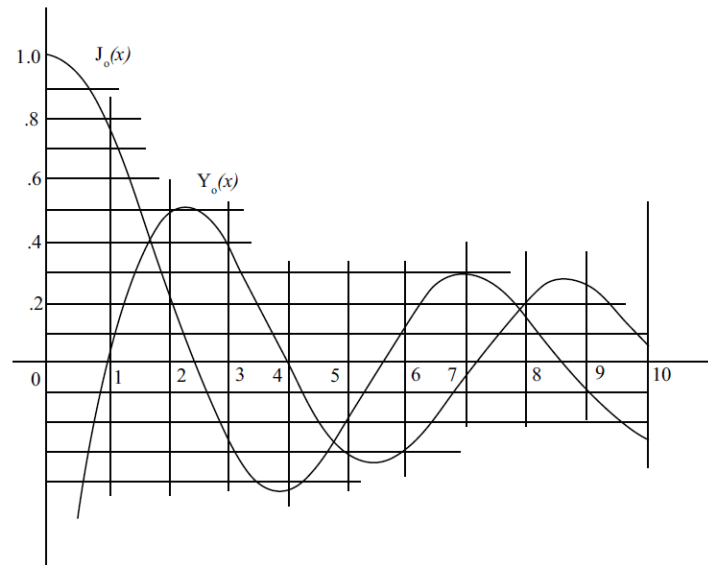
The complete solution of Bessel's equation of order zero is

$$y = AJ_0(x) + BY_0(x)$$



Bessel's functions of order zero

2-D Heat conduction in a Finite Cylinder



Bessel's functions of order zero

➤ Solution is

$$\theta = [C J_0(\lambda r) + D Y_0(\lambda r)] [A e^{-\lambda z} + B e^{\lambda z}] \quad (3)$$

Let us apply the boundary conditions

At $r = 0$, θ is finite

We find at $r = 0$, $J_0(0) = 1$, $Y_0(0) = \infty$; Therefore $D = 0$

2-D Heat conduction in a Finite Cylinder

Now

$$\theta = C J_0(\lambda r) [A e^{-\lambda z} + B e^{\lambda z}] \quad (4)$$

The next boundary condition is

at $z = 0$, $\theta = 0$ which gives

$$C J_0(\lambda r) (A + B) = 0$$

This will yield $A = -B$

Now

$$\theta = 2 C B J_0(\lambda r) \left[\frac{-e^{-\lambda z} + e^{\lambda z}}{2} \right]$$

$$\text{Or,} \quad \theta = C J_0(\lambda r) \sinh \lambda z \quad (5)$$

2-D Heat conduction in a Finite Cylinder

The following boundary condition is

at $r = R_0$, $\theta = 0$; This gives

$J_0(\lambda R_0) = 0$ by requiring that

λ assumes discrete values for which

$$J_0(\lambda_n R_0) = 0 \text{ for } n = 1, 2, 3, \dots, \infty$$

$$\theta = \sum_{n=1,2,3}^{\infty} C_n J_0(\lambda_n r) \sinh \lambda_n z \quad (6)$$

The remaining boundary condition is

$$\text{At } z = L, \quad \theta = f(r)$$

2-D Heat conduction in a Finite Cylinder

$$f(r) = \sum_{n=1,2,3}^{\infty} C_n J_0(\lambda_n r) \sinh \lambda_n L$$

$$\text{or, } f(r) = \sum_{n=1,2,3}^{\infty} A_n J_0(\lambda_n r) \quad \text{for } 0 < r < R_0$$

Now the requirement is to express this as

$$\int_0^{R_0} f(r) r J_0(\lambda_n r) dr = \int_0^{R_0} A_n r J_0^2(\lambda_n r) dr$$

We will get

$$\sum_{n=1,2,3}^{\infty} A_n = \frac{\int_0^{R_0} f(r) r J_0(\lambda_n r) dr}{\int_0^{R_0} r J_0^2(\lambda_n r) dr}$$

2-D Heat conduction in a Finite Cylinder

Here we will introduce $f(r) = T_0$. Then we can write

$$A_n = \frac{\int_0^{R_0} T_0 r J_0(\lambda_n r) dr}{\int_0^{R_0} r J_0^2(\lambda_n r) dr}$$

We will also introduce

Recurrence formulae of Bessel functions

➤ Relation-1

$$\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\frac{d}{dx} [x J_1(x)] = x J_0(x)$$

$$\int x J_0(x) dx = x J_1(x)$$

2-D Heat conduction in a Finite Cylinder

➤ Relation-2

$$\int_0^a x J_n^2(\lambda x) dx = \frac{a^2}{2} \left[\{J_n'(\lambda a)\}^2 + \left(1 - \frac{n^2}{\lambda^2 a^2}\right) J_n^2(\lambda a) \right]$$

$$\int_0^{R_0} r J_0^2(\lambda_n r) dr = \frac{R_0^2}{2} \left[(-J_1(\lambda_n R_0))^2 + J_0^2(\lambda_n R_0) \right]$$

In our case, $J_0(\lambda_n R_0) = 0$

Now in the expression for A_n , we substitute $\xi = \lambda_n r$ and get

$$A_n = \frac{\int_0^{R_0} T_0 \frac{\xi}{\lambda_n} J_0(\xi) \frac{d\xi}{\lambda_n}}{\frac{R_0^2}{2} J_1^2(\lambda_n R_0)}$$

$$\text{or, } A_n = \frac{\frac{T_0}{\lambda_n^2} \int_0^{R_0} \xi J_0(\xi) d\xi}{\frac{R_0^2}{2} J_1^2(\lambda_n R_0)}$$

2-D Heat conduction in a Finite Cylinder

$$\text{or, } A_n = \frac{\frac{T_0}{\lambda_n^2} [\xi J_1(\xi)]_0^{R_0}}{\frac{R_0^2}{2} J_1^2(\lambda_n R_0)}$$

$$\text{or, } A_n = \frac{2T_0}{\lambda_n^2 R_0^2} \frac{\lambda_n R_0 J_1(\lambda_n R_0)}{J_1^2(\lambda_n R_0)}$$

$$\text{or, } A_n = \frac{2T_0}{\lambda_n R_0} \cdot \frac{1}{J_1(\lambda_n R_0)}$$

Therefore

$$C_n = \sum_{n=1,2,3}^{\infty} \frac{2T_0}{\lambda_n R_0} \cdot \frac{1}{J_1(\lambda_n R_0)} \cdot \frac{1}{\sinh \lambda_n L}$$

2-D Heat conduction in a Finite Cylinder

Invoking the value of C_n in Eqn. (6), we get

$$\theta = \frac{2T_0}{R_0} \sum_{n=1,2,3}^{\infty} \frac{\sinh \lambda_n z}{\lambda_n \sinh \lambda_n L} \cdot \frac{J_0(\lambda_n r)}{J_1(\lambda_n R_0)} \quad (7)$$

This is the final expression for the temperature distribution at any (r, z) location in the domain.

We may recall that $T_2 - T_1 = f(r) = T_0$

$$\theta(r, z) = \frac{2(T_2 - T_1)}{R_0} \sum_{n=1,2,3}^{\infty} \frac{\sinh \lambda_n z}{\lambda_n \sinh \lambda_n L} \frac{J_0(\lambda_n r)}{J_1(\lambda_n R_0)}$$

Thank you